University College London
DEPARTMENT OF MATHEMATICS
Mid-Sessional Examinations 2011
Mathematics 1101
Wednesday 12 January 2011 2.30 – 4.30

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) State the definition of $\lim_{x \to \xi} f(x) = l$.

   (b) State what it means for the function f to be continuous at ξ.

   (c) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^3 - 7, & (x \le 2), \\ 2/x, & (x > 2). \end{cases}$$

   Prove carefully (using ϵ and δ) that f is continuous at $\xi = 2$.

   (d) Explain why f is continuous on $\mathbb{R}$.

   Hint: There is no need to use ϵ and δ in (d), if you quote the right theorems.

2. (a) State what it means for a real sequence to converge.

   (b) State the Archimedean property of the natural numbers.

   (c) Prove that every convergent sequence is bounded.

   (d) Explain why the sequence $\langle x_n \rangle$ with

$$x_n = n - \frac{1}{2^n}$$

   does not converge.

   (e) State what it means for a real sequence to be a Cauchy sequence.

3. (a) State and prove the Cauchy–Schwarz inequality.

   (b) Show that, if $x_1, x_2, \ldots, x_n$ are positive real numbers, then

$$n^2 \le (x_1 + x_2 + \cdots x_n) \cdot \left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \right).$$

   (c) Let a, b, c be positive numbers with $a + b + c = 1$. Show that

$$\left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right) \ge 8.$$

PLEASE TURN OVER

4. (a) State and prove the Intermediate Value Theorem.

(b) Let $f : [0,1] \to [0,1]$ be continuous on $[0,1]$.

Prove that for some $\xi \in [0,1]$ we have $f(\xi) = 1 - \xi$.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function for which

$$(f(x))^2 - 2e^x f(x) - 3 = 0, \quad x \in \mathbb{R}, \quad f(0) = 3.$$

Show that

$$f(x) = e^x + \sqrt{e^{2x} + 3}, \quad \forall x \in \mathbb{R}.$$

5. (a) Define e^x for $x \in \mathbb{R}$ as a series. Show that the series converges for all $x \in \mathbb{R}$.

(b) Show that the series $\sum_{n=1}^{\infty}(1 - e^{-1/n})$ diverges.

(c) Determine with explanations whether the following series converge or diverge.

$$\sum_{n=1}^{\infty} \frac{3}{2^n + 1}, \qquad \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n \cdot \ln(n)}.$$

6. (a) State the Arithmetic Mean – Geometric Mean Inequality for n non-negative numbers $a_1, a_2, \ldots, a_n$.

(b) Prove that the sequence $x_n = \left(1 + \frac{1}{n}\right)^n$ is increasing.

(c) Prove that the sequence $y_n = \left(1 + \frac{1}{n}\right)^{n+1}$ is decreasing.

(d) Show that $x_n < y_n$. Deduce that x_n is bounded above, while y_n is bounded below. Conclude that they have the same limit.

END OF PAPER